

MRC-PH-TH-11-96
q-alg/9609028

A q -oscillator Green Function

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Abstract

By using the generating function formula for the product of two q -Hermite polynomials q -deformation of the Feynman Green function for the harmonic oscillator is obtained.

PACS numbers: 03.65.Fd and 02.20.- a

September 1996

1 Introduction

q-oscillators are the most extensively studied deformed dynamical systems. They have been presented in severent different types [1]. The e-functions of these q-oscillators are expressible in terms of either the discrete or the continuous q-Hermite polynomials [2].

Although the literature on q-oscillators is very rich, the corresponding q-Green functions have not been investigated. This looks suprising when one considers the fact that the exact closed form of the Green function of the undeformed oscillator has been known for many decades [3].

It is the purpose of this note to obtain the Green function for one of the q-oscillators. The q-oscillator we deal with is the one which is solved in terms of the continuous q^{-1} -Hermite polynomials [4].

In Section II we briefly review the q-oscillator realization of Ref.4. Section III is devoted to the derivation of the q-oscillator Green function which is the deformation of the well known Feynman formula. The method of the calculation of the non-trivial $q \rightarrow 1$ limit , which is essential for arriving at the usual Feynman Green function is outlined in the Appendix.

2 A q-oscillator realization

Recently Atakishiev, Frank and Wolf introduced a simple difference realization of the Heysenberg q-algebra [4].They also studied the corresponding q-oscillator Hamiltonian and its e-functions in terms of the q^{-1} -Hermite polynomials.The q-annihilation and creation operators acting on the smooth functions $f(\xi)$ with $\xi \in (-\infty, \infty)$ are given by

$$b_q = \frac{1}{2^{1/2}q^{1/4}\beta}v(\xi)(q^{-\beta\xi}\exp(\frac{1}{2\beta}\partial_\xi) - q^{\beta\xi}\exp(-\frac{1}{2\beta}\partial_\xi))v(\xi) \quad (1)$$

$$b_q^\dagger = \frac{1}{2^{1/2}q^{1/4}\beta}v(\xi)(q^{-\beta\xi}\exp(-\frac{1}{2\beta}\partial_\xi) - q^{\beta\xi}\exp(\frac{1}{2\beta}\partial_\xi))v(\xi) \quad (2)$$

where

$$k = -\log q, \quad \beta = \frac{1}{(2(1-q))^{1/2}}, \quad v(\xi) = \frac{1}{(\cosh(k\beta\xi))^{1/2}} \quad (3)$$

and ξ is the dimensionless variable (with $\hbar = 1$)

$$\xi = \sqrt{\omega m} x. \quad (4)$$

The algebra satisfied by the operators (1), (2) is

$$b_q b_q^\dagger - q b_q^\dagger b_q = 1. \quad (5)$$

In the limit $q \rightarrow 1^-$ ($k \rightarrow 0^+$) b_q, b_q^\dagger takes the usual forms:

$$b = \frac{1}{2^{1/2}} \left(\xi + \frac{d}{d\xi} \right) \quad \text{and} \quad b^\dagger = \frac{1}{2^{1/2}} \left(\xi - \frac{d}{d\xi} \right). \quad (6)$$

The operator

$$H_q = b_q^\dagger b_q \quad (7)$$

which is self-adjoint under the inner product

$$(f, g) = \int_{-\infty}^{\infty} d\xi \overline{f(\xi)} g(\xi) \quad (8)$$

satisfies the e-value equation

$$H_q \Psi_n^q(\xi) = [n] \Psi_n^q(\xi). \quad (9)$$

Here $[n]$ is defined as usual as

$$[n] = \frac{1 - q^n}{1 - q}, \quad n = 0, 1, 2, \dots \quad (10)$$

and the e-functions are given by

$$\Psi_n^q(\xi) = \left(\frac{k}{\pi(1-q)} \right)^{1/4} \frac{q^{(n+1/2)^2/4}}{((q; q)_n)^{1/2}} (\cosh(k\beta\xi))^{1/2} \exp(-k\beta^2\xi^2) h_n(\sinh(k\beta\xi) \mid q) \quad (11)$$

h_n is the continuous q^{-1} - Hermite polynomial and $(q; q)_n$ is the q - factorial :

$$(q; q)_n = \prod_{j=1}^n (1 - q^j) \quad (12)$$

In $q \rightarrow 1^-$ limit h_n takes the form of the usual Hermite polynomial (with $\sinh(k\beta\xi) \rightarrow (\frac{1-q}{2})^{1/2}\xi$):

$$\lim_{q \rightarrow 1^-} \left(\frac{2}{1-q} \right)^{n/2} h_n(\sinh(k\beta\xi) \mid q) = H_n(\xi). \quad (13)$$

3 A “physical” q-oscillator and its Green function

Making use of the operator (7) we can write the following “physical” q-oscillator Schrödinger equation including the ground state energy :

$$(q^{1/2}b_q^\dagger b_q + [\frac{1}{2}])\Phi_n^q(\xi, t) = \zeta D_\zeta^q \Phi_n^q(\xi, t). \quad (14)$$

Here ζ is the exponential time parameter given by

$$\zeta = \exp(-i\omega t) \quad (15)$$

and $\Phi_n^q(\xi, t)$ is the time dependent wave function :

$$\Phi_n^q(\xi, t) = \exp(-i\omega(n + 1/2))\Psi_n^q(\xi) = \zeta^{n+1/2}\Psi_n^q(\xi). \quad (16)$$

The action of the q- derivative on the time dependent factor of the above wave function

$$\zeta D_\zeta^q \zeta^{n+1/2} = [n + 1/2]\zeta^{n+1/2} = (q^{1/2}[n] + [\frac{1}{2}])\zeta^{n+1/2} \quad (17)$$

exhibits the correct energy spectrum of the ”physical“ q- oscillator.

Time dependent wave function enable us to write the q- Green function for the oscillator as

$$K_q(\xi, \xi'; z) = \sum_{n=0}^{\infty} z^{n+1/2} \Psi_n^q(\xi) \Psi_n^q(\xi') \quad (18)$$

where

$$z = \bar{\zeta}\zeta' = \exp(-i\omega(t' - t)). \quad (19)$$

To execute the summation over n in (18) we recall the following generating function formula for the product of two continuous q - Hermite polynomials [5] :

$$\frac{(z^2; q)_\infty}{(z \exp(i(\theta + \phi)); q)_\infty (z \exp(i(\theta - \phi)); q)_\infty (z \exp(-i(\theta + \phi)); q)_\infty} \times \\ \times \frac{1}{(z \exp(-i(\theta - \phi)); q)_\infty} = \sum_{n=0}^{\infty} z^n \frac{H_n(\cos \theta | q) H_n(\cos \phi | q)}{(q; q)_n} \quad (20)$$

Here $(\alpha; q)_\infty$ is defined as

$$(\alpha; q)_\infty = \prod_{j=0}^{\infty} (1 - \alpha q^j) \quad (21)$$

In $q \rightarrow 1^-$ limit (see Appendix) the formula (20) is reduced to the well known summation formula for the product of two undeformed Hermite polynomials [6]:

$$\frac{1}{(1 - z^2)^{1/2}} \exp\left[-\frac{1}{1 - z^2}(z^2(\xi^2 + \xi'^2) - 2z\xi\xi')\right] = \\ \sum_{n=0}^{\infty} \frac{z^n}{2^n n!} H_n(\xi) H_n(\xi') \quad (22)$$

with

$$\xi = \left(\frac{1 - q}{2}\right)^{1/2} \cos \theta, \quad \xi' = \left(\frac{1 - q}{2}\right)^{1/2} \cos \phi. \quad (23)$$

Note that by the help of (22) one can derive the well known Feynman formula for the undeformed oscillator [3] (with $T = t' - t$)

$$K(\xi, \xi'; t' - t) = \left(\frac{m\omega}{2\pi i \sin(\omega T)}\right)^{1/2} \exp\left[\frac{im\omega}{2 \sin(\omega T)}((\xi^2 + \xi'^2) \cos(\omega T) - 2\xi\xi')\right] \quad (24)$$

from the Green function written in the wave function decomposition form [7].

To derive the q - oscillator Green function we first insert q^{-1} in place of q by recalling the relation [5]

$$\frac{1}{(q^{-1}; q^{-1})_n} = q^{1/8} q^{(n+1/2)^2/2} \frac{(-1)^n}{(q; q)_n}. \quad (25)$$

After making the required analytic continuations in θ and ϕ we arrive at

$$\begin{aligned} & E_q^{-1}\left(\frac{qz^2}{1-q}\right) E_q\left(\frac{-qz}{1-q} \exp -(\theta + \phi)\right) E_q\left(\frac{-qz}{1-q} \exp(\theta + \phi)\right) \\ & E_q\left(\frac{qz}{1-q} \exp -(\theta - \phi)\right) E_q\left(\frac{qz}{1-q} \exp(\theta - \phi)\right) \\ & = \sum_{n=0}^{\infty} \frac{h_n(\sinh \theta | q) h_n(\sinh \phi | q)}{(q; q)_n} q^{(n+1/2)^2/2} z^n \end{aligned} \quad (26)$$

which is the formula suitable to our q^{-1} - Hermite polynomials. The q - exponentials employed in the above equation are given in terms of the $n \rightarrow \infty$ limit of the q - factorials as [5]

$$E_{1/q}(-x) = E_q^{-1}(x) = ((1-q)x; q)_{\infty}. \quad (27)$$

When we introduce the formula of (26) into (18) we obtain the final form of the q -oscillator Green function:

$$\begin{aligned} K_q(\xi, \xi'; z) &= q^{-1/8} \left(\frac{2k}{\pi}\right)^{1/2} \beta (\cosh(k\beta\xi) \cosh(k\beta\xi'))^{1/2} \exp(-k\beta^2(\xi^2 + \xi'^2)z^{1/2}) \\ & E_q^{-1}\left(\frac{qz^2}{1-q}\right) E_q\left(\frac{-qz}{1-q} \exp(-k\beta(\xi + \xi'))\right) E_q\left(\frac{-qz}{1-q} \exp(k\beta(\xi + \xi'))\right) \\ & E_q\left(\frac{qz}{1-q} \exp(-k\beta(\xi - \xi'))\right) E_q\left(\frac{qz}{1-q} \exp(k\beta(\xi - \xi'))\right) \end{aligned} \quad (28)$$

By the process sketched in the Appendix the above equation is reduced to the Feynman formula of (24) in $q \rightarrow 1^-$ limit.

In $T \rightarrow 0$ ($z \rightarrow 1$) limit we distinguish two cases :

(i) For $\xi \neq \xi'$ by the virtue of the first exponential

$$E_q^{-1}\left(\frac{qz^2}{1-q}\right) = (qz^2; q)_{\infty} = \prod_{n=1}^{\infty} (1 - q^n z^2). \quad (29)$$

we have

$$\lim_{T \rightarrow 0} K_q(\xi, \xi'; z) = 0. \quad (30)$$

(ii) For $\xi = \xi'$ on the other hand by the virtue of the 1st, 2th and 3th expanantioles we have

$$\begin{aligned} & \lim_{z \rightarrow 1} E_q^{-1}\left(\frac{qz^2}{1-q}\right) E_q\left(\frac{-qz}{1-q}\right) E_q\left(\frac{-qz}{1-q}\right) \\ &= \lim_{z \rightarrow 1} \prod_{n=0}^{\infty} \frac{(1 - q^n z^2)}{(1 - q^n z)(1 - q^n z)}. \end{aligned} \quad (31)$$

The n=1 factor in the above equation contributes a singularity of

$$\lim_{z \rightarrow 1} \frac{1}{1-z} \quad (32)$$

type.

It is easy to conclude then that the Green function (28) behaves as the δ -function $\delta(\exp(k\beta\xi) - \exp(k\beta\xi'))$ in $T \rightarrow 0$ ($z \rightarrow 1$) limit.

Acknowledgement. We thank Ö.F.Dayi for reading the manuscript.

APPENDIX

Using the definitions in (21) and (23) we can rewrite (20) as

$$\begin{aligned} & \prod_{j=0}^{\infty} (1 - z^2 q^j) (1 - z^2 q^{2j})^{-2} (1 - (1-q)A_j)^{-1} = \\ & \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} H_n\left(\left(\frac{1-q}{2}\right)^{1/2} \xi \mid q\right) H_n\left(\left(\frac{1-q}{2}\right)^{1/2} \xi' \mid q\right) \end{aligned} \quad (A.1)$$

with

$$A_j = 2 \frac{z q^j \xi \xi' (1 + z^2 q^{2j}) - (\xi^2 + \xi'^2) z^2 q^{2j}}{(1 - z^2 q^{2j})^2} \quad (A.2)$$

$q \rightarrow 1^-$ limit of the right hand side (r.h.s) of (A.1) (with $(q; q)_n = (1-q)^n [n]$) is

$$\lim_{q \rightarrow 1^-} (r.h.s) = \sum_{n=0}^{\infty} \frac{z^n}{2^n n} H_n(\xi) H_n(\xi') \quad (\text{A.3})$$

Let us take the logarithm of the left hand side (l.h.s.) of (A.1) :

$$\log(l.h.s.) = \sum_{j=0}^{\infty} [\log(1 - z^2 q^j) - 2 \log(1 - z^2 q^{2j})] - \sum_{j=0}^{\infty} \log(1 - (1 - q) A_j) \quad (\text{A.4})$$

Expanding the logarithm function into the series (for $|z| \leq 1$) the first two terms in the above equation can be written as

$$\begin{aligned} \sum_{j=0}^{\infty} [\log(1 - z^2 q^j) - 2 \log(1 - z^2 q^{2j})] &= \sum_{k=1}^{\infty} \frac{z^{2k}}{k(1 + q^k)} \\ &= -\log(1 - z^2)_{q^2}^{1/2} \end{aligned} \quad (\text{A.5})$$

with [5]

$$(1 - z)_q^a = \psi_{1,0}(q^{-a}, q; z) \quad (\text{A.6})$$

Then (l.h.s.) can be rewritten as

$$(l.h.s.) = \frac{1}{(1 - z^2)_{q^2}^{1/2}} \exp(-F(\xi, \xi', z | q)) \quad (\text{A.7})$$

where

$$F(\xi, \xi', z | q) = \sum_{j=0}^{\infty} \log(1 - (1 - q) A_j) \quad (\text{A.8})$$

From (A.2) we see that the functional sequence $|A_j|$ for any value of z, ξ and ξ' (except the case $z=1$) decreases :

$$|A_0| > |A_1| \dots > |A_n| \dots \quad (\text{A.9})$$

Let us consider the zeroth term of the sequence (A.9) $|A_0|$ and fix the value of z (with $z \neq 1$). It is clear that there exist $n \in \mathbb{N}$ such that

$$(1 - q_i) |A_0| < 1 \quad i > n. \quad (\text{A.10})$$

where q_i is the sequence: $\lim_{i \rightarrow \infty} q_i = 1^-$. By the virtue of (A.9) the functions $(1 - q_i) | A_j |$ satisfy the condition (A.10) too. Thus the logarithm function (A.9) can be expanded in the Taylor series in $q \rightarrow 1^-$ limit as

$$\lim_{q \rightarrow 1^-} F(\xi, \xi', z | q) = \lim_{q \rightarrow 1^-} \sum_{k=1}^{\infty} \frac{(1-q)^k}{k} \sum_{j=0}^{\infty} (A_j)^k \quad (\text{A.11})$$

In the above expression only the $k=1$ term survives:

$$\lim_{q \rightarrow 1^-} F(\xi, \xi', z | q) = \lim_{q \rightarrow 1^-} (1-q) \sum_{j=0}^{\infty} A_j \quad (\text{A.12})$$

After expanding the denominator of A_j into the power series we arrive at

$$\lim_{q \rightarrow 1^-} F(\xi, \xi', z | q) = \frac{z^2(\xi^2 + \xi'^2) - 2z\xi\xi'}{1 - z^2} \quad (\text{A.13})$$

From (A.8) and (A.14) we get

$$\lim_{q \rightarrow 1^-} (l.h.s.) = \frac{1}{(1 - z^2)^{1/2}} \exp\left[-\frac{z^2(\xi^2 + \xi'^2) - 2z\xi\xi'}{1 - z^2}\right] \quad (\text{A.14})$$

which together with (A.3) establishes the desired limit of (22).

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